Many pathways towards “Excellence” in Singapore mathematics education

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This paper presents a snapshot of Singapore’s journey towards excellence in mathematics education by examining the role of the traditional notion of mathematics competition and other competitive activities. It could be seen using the context of mathematics competition that the notion of “excellence” has evolved over time. Excellence as a high standard for individuals to achieve or as a set of obstacles for individuals to pit against the norm has been gradually broadened to include excellence as an internal goal for an individual to achieve, and even excellence as a goal for the mathematics education landscape.

The Singapore Education System

Since the independence of Singapore in 1965, developing a robust education system has been the focus of the nation. Recognising that Singapore had no hinterland or natural resources, the young nation had since been striving towards building an efficient, universal education system to fulfil the role of economic development and social cohesion in Singapore beginning with the visionary leadership of the prime minister Mr Lee Kwan Yew (NTU President, 2015). The importance of education to Singapore has continued to be emphasised by the Singapore politicians. In 2001, the then prime minister, Mr Goh Chok Tong, during a Teachers’ Day Rally on 31 August acknowledged that the “skills and resourcefulness of our people” are pivotal for the nation’s survival.

The education system in Singapore has been recognized as of a high quality. A speech in 1999 by the Education Minister, Teo Chee Hean, that Singapore has “no failing schools, only good schools, and very good schools…” (a full speech is provided in Ang, 2006) is a testimony to this. Further, the performance of Singapore students in the Trends in International Mathematics and Science Study (TIMSS), the largest international comparative study of student achievement in the two subjects, and the Programme for International Student Assessment (PISA) is among the top of all the participating nations. The students’ performance in these two international comparative studies is usually taken among the indicators of the quality of a nation’s education system.

With regard to the performance of the top elite students, Singapore has also performed very well in the International Olympiads of the three sciences (Physics, Chemistry and Biology) and Informatics (Lee, n.d.). Beginning from 2011, Singapore has also emerged in the first ten positions in the International Mathematical Olympiad (IMO), according to the information provided in the official website of the IMO.

The stellar performance of students’ in TIMSS and PISA, and the prestigious Olympiads, are signs of the ongoing pursuit of excellence in the Singapore education landscape. The term “excellence” is found in the Singapore education documents. For example, the term “excellence” is found in the Singapore school management system, which is known as the *School Excellence Model*. Schools are being empowered to develop themselves into “excellent schools” based on the appraisal system of the School Excellence Model. Such a school appraisal strategy has been shown to have a significant impact on student performance (e.g., Huang et al., 2019). To achieve professional excellence among practicing teachers, centres of Teaching and Learning Excellence were set up in 2015 to provide them

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with up-to-date professional development so that practicing teachers bring back to their respective schools up-to-date teaching and learning strategies to impact the quality of teaching and learning there (Academy of Singapore Teachers, n.d.). Prior to that, within each zone in Singapore, a Centre of Excellence for mathematics had been set up as a platform for promoting the professional growth of mathematics teachers in that zone (Chua, 2009).

Singapore's pursuit of excellence can be understood by the social-cultural context of Singapore. The Singapore society has been engineered to embrace "a pragmatic and competitive national paradigm grounded in economic rationalism" (Ang, 2006, p. 1). Lessons learnt along the road to the nation's independence and the nation's vulnerability as a nation without resources are two factors that shaped the development of a competitive mindset (Cooper, 2001; Lee, 1998 cited in Ang, 2006).

Much has been done in Singapore beyond the nation's visible pursuit of "excellence" in the Singapore education landscape discussed above. The nation has traversed a long journey in shaping its own definition of excellence of education at the various levels of the society. With reference to mathematics education in particular, how the notion of "excellence" has evolved is discussed in this paper using the illustration of mathematics competition.

A two-pronged approach to excellence: Grounds up and top down approaches

The word "excellence" can be roughly understood as "exceptionally good and of superior quality" (Lierse, 2018). Based on this notion of excellence, an excellent education system refers to one that is exceptionally good and of superior quality. This vague notion of excellence in education has been operationalised. According to the European Network of Education Councils (EUNEC), excellence in education should transcend the "quality control" or even the benchmarking of education systems to *identifying, developing and intensifying talents* within the education system (EUNEC, 2012).

Two lines of effort in the pursuit of excellence in Singapore mathematics education can be discerned: the approach to excellence from (1) the grounds up; and (2) the top down. The grounds up approach towards excellence in mathematics includes the efforts by educational institutions and professional bodies to identify and develop mathematical talents; the top down approach refers to policies that impact the systemic level in achieving excellence. In this paper, we focus the discussion primarily on the grounds up approach; and briefly discuss the top down approach. A detailed discussion of the latter has been presented in Toh (in press), hence will not be further elaborated in this paper.

The notion of "excellence" in the case of mathematics competitions

In discussing "excellence" in mathematics, the idea of competitive activities as opportunities to pit against the norms (Franks, 1996) is readily forthcoming to the mind. Mathematics competition is part of the grounds up approach initiated by the local mathematics community. It is recorded that the first national level mathematics competition in Singapore emerged prior to its independence in 1956 by the Singapore Mathematical Society, which was founded at that time. Note that the first IMO was first launched three years after that in 1959 in Romania. Following the launch of the first mathematics competition in Singapore and the IMO, various other mathematics competitions at the national and school levels have started in the decades that followed. The first mathematics competitions were organised for upper secondary and high school students. This age group was the target as it was the participating age group of students for the IMO. This corresponded to identifying and nurturing of mathematical talents, and started a systematic process of identifying and developing talents for the IMO.

Subsequently, other competitions were organized for students of younger age groups at the primary levels. Not only that, the mathematics competitive activities also scaled out from the top elite group of students to the vast majority of the student population. A full description of the emergence of various mathematics competitions and their evolution can also be found in Toh (in press). Alongside identifying, developing and nurturing mathematical talents, the pursuit of excellence in identifying and nurturing talents had also broadened to include selecting potential students from other Singapore mainstream schools. The evolution of the mathematics competitions in Singapore can be traced to at least three phases: (1) identifying and nurturing mathematical talents; (2) popularizing mathematics among a wider student population beyond potential competition contestants; and (3) aligning to the Singapore mathematics education.

Phase 1: Identifying and nurturing mathematical talents

This phase began with the first mathematics competition organized by the Singapore Mathematical Society in 1956, to around the early 1990s. In this phase, the key objective of identifying and nurturing mathematical talents could be seen as aligning to the selection of the best among the mathematical talents to represent the nation in the IMO and other prestigious international mathematics competitions. This phase corresponded to the pursuit of excellence as reaching the highest possible standard in mathematics.

Phase 2: Popularizing mathematics among a wider student population beyond potential competition contestants

This phase approximately corresponded to the period from 1990 to 2010. Starting from 1990, mathematics competition of the primary school students was launched and in 1994, the Singapore Mathematics Olympiad (SMO), the most prestigious mathematics competition at the national level, launched the Junior Section for lower secondary students in addition to the usual Senior Section (for upper secondary students) and Open Section (for the pre-university students).

Phase 2 was characterised by the effort of the mathematics community to popularize mathematics to a much wider student population, in addition to identifying and nurturing mathematical talents. In 1994, in the collection of challenging mathematics problems collated from the various interschool and national mathematics competitions published by the Singapore Mathematical Society, it was stated that the objective of the collection of problems was to “inspire in its readers the desire to learn more about mathematics” (Singapore Mathematical Society, 1994, p. ii). Various compilation of competition questions for different student levels were subsequently published with the objective to “stimulate interest and develop prowess in mathematics among students in the primary schools of Singapore” (The Chinese High School, 2003, p. ii), or to “instil a love for and to generate interest in Mathematics amongst Primary school students” (National University of Singapore High School of Math & Science, 2007, p. i). This phase showed a broadened notion of excellence as individualised; reaching an individualised peak of excellence is a worthy goal.

Phase 3: aligning to the Singapore mathematics education

The third phase began in the early 2010s, and this phase was characterised by a conscious effort of the mathematics communities in aligning the mathematics competition to the school mathematics curriculum, in addition to the objectives of the previous two phases. In the preface of the compilation of the past year SMO questions, the compilers commented that "We align the SMO more closely to the school curriculum … there will be a considerable number of questions in Round 1 [the section that all contestants will attempt] of each section which are based on the school curriculum…" (Ku et al., 2016, 2017, 2018, p. ii). The mathematics competition questions no longer exclusively contained the extremely challenging questions which are beyond the reach of the general student population. A considerable number of the mathematics competition questions were based on the contemporary school mathematics curriculum, although many of these questions require a creative use of the mathematical techniques taught in school mathematics. The subtle difference between Phases 2 and 3 is that while both phases saw a similar effort to reach out to a wider range of students, there was a visible effort to align to the school mathematics curriculum in Phase 3, thereby possibly impacting the classroom mathematics instruction. The notion of excellence in this phase has expanded beyond individual peak of excellence, to encompass excellence in the teaching and learning processes for all teachers and students.

Mathematics competition questions beyond competition

As discussed above, in Phase 3, the link between mathematics competitions and the school mathematics curriculum has become explicit. The intention of the local mathematics communities to align the prestigious mathematics competitions to the local school mathematics syllabuses had enlarged the functions of the mathematics competition questions. More competition questions were then made accessible and were being accessed by the general student population. A larger student population had then the opportunity to challenge themselves with the mathematics competition questions which were within their capacity, and to reflect on the school mathematics content that they have learnt.

Mathematics competition questions have also been valued because of the affordances of these items in the preservation of the "old" mathematical techniques within the contemporary mathematics syllabuses. These techniques have been de-emphasised in the curriculum due to an increased emphasis on technology in the school curriculum (Toh, 2015). Many of the problems that require these "old" mathematical techniques epitomise a high degree of creativity in the use of more delicate mathematical techniques (without resorting to technology). This is still relevant to the Singapore mathematics curriculum, which emphasises mathematical problem solving. Illustrations 1 and 2 are exemplars of this category of problems, which could serve to motivate more students to acquire creative mathematical techniques for the mathematical content which is found in the current syllabuses and appreciate the nature and beauty of mathematics.

Illustration 1: Simplify $144\left(\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}\right)$.

(A modified item from a typical genre of the SMO questions on simplifying surds without the use of calculating tools)

Illustration 2: Which of the following numbers is largest?

(A) $\sqrt{10}-\sqrt{9}$
(B) $\sqrt{20}-\sqrt{19}$
(C) $\sqrt{30}-\sqrt{29}$
(D) $\sqrt{40}-\sqrt{39}$
(E) $\sqrt{50}-\sqrt{49}$

(A modified item from a typical genre of questions on comparing the magnitude of surds without the use of calculating tools)

The solution of Illustration 1 can be obtained indirectly by considering the square of the given expression. A careful application of the rules of surds will result in a perfect square, for which the square root of the square number yields the answer. Illustration 2 can be solved by considering the process of irrationalising each of the five surdic expressions, and comparing the five fractions which have equal numerator. Such problem solving strategies which lead to elegant solutions are not stressed in the mainstream curriculum, as the use of calculating tools renders such strategies unnecessary. This is further hindered by the provision of calculators for all high-stake national mathematics examinations.

Other competition questions engage the solvers to think more deeply and reflect on the usual misconceptions that students have in applying algorithmic procedures (exemplified by Illustrations 3 and 4 below). Such items are atypical of high-stake national examinations. Illustration 3 challenges the solver to re-think their usual understanding of solving an algebraic equation in relation to the process of like-terms in both sides of an equation. This makes them re-think of the equivalence of the two equations, and easily relates to the big idea of Equivalence in mathematics. Illustration 4 invites the solver to examine the common misconception that $\sqrt{a^2} = a$ for all real values of a. The preservation of such items within the existing mathematics competitions is an indicator of the effort to emphasise the metacognitive aspect of problem solving, which is stressed in the syllabuses.

Illustration 3: How many real numbers x satisfy the equation $\frac{x^2-x-6}{x^2-7x-1} = \frac{x^2-x-6}{2x^2+x+15}$?

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Illustration 4: Let $a < 0$. Find $\sqrt{a^2} + \sqrt{(1-a)^2}$.

(A) 1 (B) -1 (C) $2a - 1$ (D) $1 - 2a$ (E) None

Some mathematicians lament that the mathematics curriculum today is far from the level of difficulty of that in the 1980s (e.g., France & Andzans, 2008). The various mathematics competitions, with their unofficial "syllabuses" for the competition and the lack of provision of allowing calculating devices, serve to preserve many of the elegant mathematical content which were otherwise not emphasised in the contemporary syllabuses. With the trend of increasing student participation in the various local mathematics competitions, many of these mathematical questions with elegant solutions are kept alive but are downplayed in the mainstream school curriculum.

A further step to popularize competition-type of mathematics problems is found in the contemporary mathematics textbooks which have been approved by the Singapore Ministry of Education (MoE) for schools. Under the paradigm of differentiated instruction, the inclusion of tiered practice tasks in the textbooks has resulted in the inclusion of many of such competition-type questions. The ready availability of such questions, usually classified under the section "challenging questions" (or similar classification of tasks to the same effect), is a further step to engage all students to challenge themselves in higher level mathematical thinking. This is especially important for the students who might not participate in mathematics competitions.

The notion of excellence in mathematics competition has also expanded to influence professional development of mathematics teachers as well. From the author's first-hand experience in working directly with practicing teachers in the Singapore schools in several of the teacher professional development activities, many of the challenging mathematics competition questions have provided opportunity for teachers to identify the "blind spots" in their own knowledge of mathematics. It is common knowledge that mathematical content knowledge which is not frequently tested in the high-stake national exams tends to be out of a teacher's attention. The occurrence of such items in the various mathematics competitions could also bring a teacher to reflect on the content essential for classroom teaching. Some of these items have been incorporated into professional development courses for teachers. We consider one example in the Singapore Additional Mathematics syllabus using illustration 5 below, which is an item adapted from a past competition question (year unidentified). This item brought out several interesting discussions among the author and some secondary school teachers about logarithms.

Illustration 5: Find the value of $9^{2\log_9 5}$ without the use of calculator
(Adapted from a past year competition question in Singapore Mathematical Olympiad)

Although the following rule of logarithm is common knowledge for most students and teachers,

$$\log_a a^x = x$$

this rule is usually understood by most teachers and students in the usual computational sense as a procedural rule:

$$\log_a a^x = x \log_a a = x.$$

The following rule, which is a counterpart of the above rule of logarithm,

$$a^{\log_a x} = x$$

is less well-known among students and teachers. Although both rules involve the composition of a function and its inverse (i.e., the exponential function and the logarithmic function), the first rule can be easily algorithmised as "shifting the power of a logarithm down" while it is recognisably more difficult to proceduralise the second rule. The occurrence of items such as Illustration 5 reminds the teachers of the importance of the notion of the composition of a function and its inverse, rather than a pure utility of logarithms as a tool for conversion to exponential function (Kenny et al., 2013). This is an important alert to teachers that the concept of function underpins most mathematical concepts in the syllabuses, although explicit knowledge of functions and their composition are not required for the national examinations in the secondary school mathematics syllabuses (MoE, 2018).

Mathematics competition questions and problem solving

A further stage in utilizing the mathematics competition questions is in adapting them for teaching mathematical problem solving to *all* secondary mathematics students (that is, problem solving is not only reserved for the elite few, but for the whole student population). As it is well-known, mathematical problem solving is the heart of the Singapore mathematics curriculum. In New Zealand, Holton (2010) introduced mathematical problem solving processes to IMO students through imparting them the mathematical content knowledge on discrete mathematics. Motivated by this approach, a similar effort in mathematics education research in Singapore emerged in the late 2000s to the early 2010s.

The new interpretation of problem solving using the science practical paradigm (i.e. problem solving to mathematics is in the same way as science practical to science) in an effort to make problem solving accessible for *all* students, and to illustrate to teachers how an authentic problem solving lesson can be enacted in the mathematics classroom. Broadly speaking, problem solving lessons in mathematics should be treated as science practical lessons in science, and the role of teachers is to facilitate the students' experience of the entire problem solving process (Toh et al., 2008). This initiative was introduced in recognition of the fact that most school mathematics teachers might not have taught students problem solving to the true sense of its spirit as proposed by Pólya (1945). This approach to teaching problem solving is contrary to many teachers' usual classroom practice in "routinizing the problems" into exercises for the students.

A detailed discussion on the conceptualisation of the science practical paradigm, proposal on how problem solving lessons could be enacted in the mathematics classrooms, and the reports of the various experiment schools about their successes and challenges in enacting a problem solving lesson have been discussed (Leong et al., 2013; Toh et al., 2008). In the problem solving lessons, authentic problems that could highlight the various problem solving stages must be selected as the vehicles for teaching problem solving. As such, competition questions become suitable choice of questions for the teaching of problem solving. Illustrations 6 and 7 appended below are two exemplars of competition-type questions which have been used for teaching authentic problem solving.

Illustration 6: Find the last digit of 13^{77}.

Illustration 7: Find the last digit of $1962^{2009} + 2009^{1962}$.

The content of the two exemplars above is on Elementary Number Theory, which is not taught in the Singapore school mathematics curriculum. As such, these problems will be "non-routine" to most students – one of the two criteria to qualify as a "problem" (Toh et al., 2008). However, the content of these two questions are easily understandable even for a primary school student. Hence, these problems can be used as authentic problems that can serve to reinforce and illuminate the various problem solving heuristics, and can "force" students to acquire problem solving processes (in this case, looking for patterns and making conjectures for illustration 6, and, in addition, looking for sub-goals in illustration 7). In short, this type of problems is realistic enough for students to experience authentic problem solving by experiencing all the Pólya stages of problem solving.

Mathematics Competitive Activities beyond the Traditional Competition

Mathematics competitive activities have transcended the confines of the common notion of paper-and-pencil tests by the traditionalists. Some talents in mathematics and high-achieving mathematics students may be more inclined towards other forms of competitive mathematics activities, such as collaborative problem solving activities involving real world problems, or engaging in authentic mathematics research with professional mathematicians, are among the competitive activities that are designed to capture the various talents in mathematics. The biennial event of the Singapore International Mathematical Challenge is organised to provide opportunity for students to work collaboratively with their peers in solving real-world problems by making use of available technological tools and information. To develop young research mathematicians, opportunities are provided for students to work on mathematics research projects with professional mathematicians beyond their schools. The annual Singapore Mathematics Project Festival is a platform for students to showcase the fruits of their research to their contemporaries and other mathematicians. More details of alternative competitive mathematics activities are described in Toh (in press) and will not be elaborated.

In an effort to engage an even wider spectrum of students in mathematics competitive activities, the Singapore Mathematical Society has initiated a new series of mathematics essay competition, an annual event that aims to expose the participating individuals to an identified mathematical topic and to encourage the participants to articulate mathematics through the exposition on the topic (Singapore Mathematical Society, 2021). This further widens the group of students who might not be inclined to the modes of competitive mathematical activities described previously. In addition to sharpening an individual's thinking and reasoning, this activity encourages the participants to communicate mathematics precisely, clearly and logically. It is aligned to the latest emphasis in the Singapore mathematics curriculum on communication in mathematics (Kaur & Toh, 2012).

Achieving Excellence at the Systemic Level

At the systemic level, the pursuit for "excellence" has transcended the notion of a unique peak of excellence understood by the traditionalists' view. The notion of excellence has now been interpreted as the existence of many peaks, and even a peak for each student, in order to encompass excellence for every individual. The systemic effort in the pursuit of excellence can be seen to be guided by the dual objectives of enabling students of different capacities to define and reach their own peak of excellence (Shanmugaratnam, 2006) and, "lifting the bottom but not capping achievements and limiting opportunities at the top…" (Ong, 2018).

The notion of not capping achievements and limit opportunities at the top is best epitomized by the education system in identifying and nurturing talents in various way, and depicts a concerted effort by the MoE in stretching excellence to the fullest potential among an individual. The holistically talented students are identified early at the upper primary level and offered an opportunity to the Gifted Education Programme within the Singapore education programme. This specialized programme for the gifted individuals (defined as individuals who form the top 1% of the top performing students) continues to be supported by school-based gifted education programme found in selected secondary schools.

Specialized schools have been set up for students who are specifically talented in a specific discipline. In particular, the NUS High School of Science and Mathematics has been specifically set up for students who are specifically inclined towards mathematics and sciences. In this specialized school, students are not bound by the high-stake national examinations at the end of the high school as the scope of the national exams capped the learning of the students. In addition, students in this school are given the opportunity to read a subject at the undergraduate level and to even do a research project at the higher secondary levels. Under the supervision by their teachers or mathematics professors, the research work carried out by the student approximates the research work of a professional mathematician.

Another movement in the Singapore education system to move towards stretching all students' potential to the fullest is the recent introduction of subject-based banding of mathematics (and three other subjects), with the ultimate goal of pushing for subject-based banding for all subjects at the primary and secondary school education. This movement can be seen to be modelled after the pre-university education system in which the students can read each academic subject at a level that is suitable for them. Under this opportunity, all students will have the opportunity to be stretched in all disciplines according to their capacity and inclination.

Conclusion

The journey towards excellence in education is best summarized by the speech of the then Minister of Education, Mr Heng Swee Keat, during his interview with the Straits Times on 22 August 2015. Mr Heng commented that the pursuit of excellence should be "part of Singapore's DNA", but stressed the need to "broaden the definition of excellence and to recognise everyone for achieving his personal best" (The Straits Times, August 22, 2015). Even within mathematics education, it is clearly evident that Singapore is moving towards "a mountain range of excellence, not just one peak, to inspire all our young to … climb as far as they can." (Shanmugaratnam, cited in Lee et al., 2008).

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